Internet Appendix for: "The Cryptocurrency Participation Puzzle"

RAN DUCHIN, DAVID H. SOLOMON, JUN TU, and XI WANG

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This appendix presents our methods and estimates from robustness tests and extensions. Section 1 discusses the Bayesian methods used to calculate the posterior distribution. Section 2 discusses robust portfolio choice methods used to consider model misspecification. Section 3 provides estimates from robustness tests and extensions.

1 Bayesian Methods

In addition to the basic portfolio choice setup described previously, we now describe the mechanics of calculating the posterior distributions of the key parameters. Assume the (excess) returns of the N risky asset follow the following data generating process (DGP):

$$r = \mu + \epsilon, \ \epsilon \sim N(0, V) \tag{1}$$

In our study, there are $N = 6$ assets, namely: the cryptocurrency asset-rf, mkt-rf, SmB, HmL, UmD and MSCIexUS-rf.

The informative priors on Mean μ and Covariance V are given by:

$$\mu \sim N(\mu_0, \Omega V \Omega), \ \Omega = \begin{bmatrix} \rho_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \rho_N \end{bmatrix} \tag{2}$$

$$V \sim IW(V_0, h) \tag{3}$$

Following the standard empirical Bayesian approach, the prior on the mean μ, denoted as μ_0, is set to the sample mean in the pre-period preceding our portfolio choice. Intuitively, when this initial sample period is shorter, the investor would assign a weaker prior since there is less data to pin down the prior. Hence, we set ρ^2 as the inverse of the number of months in the initial sample period. This way, the shorter the initial sample data is, the larger the value of ρ is, and consequently the weaker the prior is. As $1/\rho^2$ corresponds to the strength of the prior mean, we set $1/\rho^2$ for Mkt-rf, SMB, HML, UMD as 1042, since we have 1042 months of pre-sample data for these assets. We set $1/\rho^2 = 520$ for MSCIexUS, as we have 520 months of data for it. Since we do not have pre-sample data for cryptocurrency assets, we select different values for $1/\rho^2$ to reflect different prior strengths. In the baseline case, we set $1/\rho^2 = 120$, which corresponds to an initial cryptocurrency sample data of 10 Years. For the flat prior, we set $1/\rho^2 = 1e-6$.

Moreover, when a risky asset i is more volatile, there is more parameter uncertainty about its mean μ_i. Hence, when the variance σ_i of the returns on a risky asset i is higher, we assume a weaker prior on μ_{i0} to reflect the higher parameter uncertainty.

Similarly, following the standard empirical Bayesian approach, V_0 is based on the sample covariance matrix using pre-period sample data. We set $V_0 = (h-k)V_{prior}$, where $V_{prior} \in R^{k\times k}$ and is represented by

$$V_{prior} = \begin{bmatrix} \sigma^2_{cry} & 0 & 0 & 0 & 0 \\ 0 & & & & \\ 0 & & V_{factors} & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} \tag{4}$$

where $V_{factors}$ is the sample covariance matrix of the five factors, namely, mkt =Mkt-rf, SMB, HML, UMD and MSCIexUS-rf based on 520 months of data prior to our portfolio choice analysis. Then, for these five factors, the expected covariance matrix is the same as the pre-period sample covariance matrix. In the baseline case, we set σ^2_{crypto} as $150, 170, 625$ times σ^2_{mkt} for the returns on Bitcoin (BTC), the value-weighted crypto market portfolio (VW100) and the equal-weighted crypto market portfolio (EW100), respectively. These ratios are based on the empirical sample ratios between the variance of the cryptocurrency assets and the stock market variance σ^2_{mkt}. We consider alternative priors on the variances of cryptocurrency assets in robustness tests.

Furthermore, in the baseline case we set the prior correlation between cryptocurrencies and the market portfolio as zero. We do so because the correlation was roughly zero in the decade following the introduction of Bitcoin in 2009. For robustness, we re-estimate the analyses using alternative correlation priors.

$$V_{prior} = \begin{bmatrix} \sigma^2_{cry} & \rho\sigma_{cry}\sigma_{mkt} & 0 & 0 & 0 \\ \rho\sigma_{cry}\sigma_{mkt} & & & & \\ 0 & & V_{factors} & & \\ 0 & & & & \\ 0 & & & & \end{bmatrix} \tag{5}$$

Using the above priors, we derive the posterior distribution. First, we derive the following density functions for the priors:

$$f_0(\mu|V) \propto |\Omega V \Omega|^{-\frac{1}{2}} \exp\Big\{ -\frac{1}{2}[\mu-\mu_0]' \frac{1}{\Omega} V^{-1} [\mu-\mu_0] \Big\}$$
$$p_0(V) \propto |V|^{-\frac{h+n+1}{2}} \exp\Big\{ -\frac{1}{2} tr\left[V_0 V^{-1}\right] \Big\}$$

Second, we derive the posterior joint distribution of μ, V proportional to $f(R|\mu,V)f_0(\mu|V)f_0(V)$,

$$\begin{aligned} f(\mu, V|r_1,\dots,r_T) \propto\ & |\Omega V \Omega|^{-\frac{1}{2}} \exp\Big\{ -\frac{1}{2}[\mu-\mu_0]' \frac{1}{\Omega} V^{-1} \frac{1}{\Omega} [\mu-\mu_0] \Big\} \\ & \times |V|^{-\frac{h+n+1}{2}} \exp\Big\{ -\frac{1}{2} tr\left[V_0 V^{-1}\right] \Big\} \\ & \times |V|^{-\frac{T}{2}} \exp\Big\{ -\frac{1}{2} \left[\sum_{t=1}^{T} (r_t-\mu)' V^{-1} (r_t-\mu) \right] \Big\} \end{aligned}$$

where $\frac{1}{\Omega}$ stands for Ω^{-1}.

To implement Gibbs Sampling, we derive the posterior conditional distribution $f(\mu|V,R)$ and $f(V|\mu,R)$ as follows

$$\begin{aligned} f(\mu|(r_1,\dots,r_T),V) \propto\ & \exp\Big\{ -\frac{1}{2}[\mu-\mu_0]' \frac{1}{\Omega} V^{-1} \frac{1}{\Omega} [\mu-\mu_0] - \frac{1}{2}\sum_{t=1}^{T}(\mu-r_t)'V^{-1}(\mu-r_t) \Big\} \\ \propto\ & \exp\Big\{ -\frac{1}{2}\left[\mu' \left(\frac{1}{\Omega} V^{-1} \frac{1}{\Omega} + T V^{-1} \right) \mu - 2 \left(\mu_0' \frac{1}{\Omega} V^{-1} \frac{1}{\Omega} + (\sum_t r_t)' V^{-1} \right)' \mu \right] \Big\} \end{aligned}$$

Therefore, conditional on V, μ is normally distributed, with the following mean:

$$\left(\frac{1}{\Omega}V^{-1}\frac{1}{\Omega}+TV^{-1}\right)^{-1}\left(\frac{1}{\Omega}V^{-1}\frac{1}{\Omega}\mu_0+V^{-1}(\sum_t r_t)\right)=\left(\frac{1}{\Omega}V^{-1}\frac{1}{\Omega}+TV^{-1}\right)^{-1}\left(\frac{1}{\Omega}V^{-1}\frac{1}{\Omega}\mu_0+TV^{-1}\bar{r}_T\right)$$

When $\Omega = \rho I$, this equation collapses into:

$$\frac{T}{T+1/\rho^2}\bar{r}_T+\frac{1/\rho^2}{T+1/\rho^2}\mu_0 \tag{6}$$

Note that $1/\rho^2$ intuitively measures how much investors trust the prior μ_0, assuming they observe data with $1/\rho^2$ periods and mean μ_0. As $\rho \to 0$, investors will trust μ_0 as if it has been observed over an infinitely long period, and consequently the sample average becomes the estimator for the population mean. In addition, μ has a conditional covariance matrix $\left(\frac{1}{\Omega}V^{-1}\frac{1}{\Omega}+TV^{-1}\right)^{-1}$.

Moreover, for the posterior distribution of V, we have that $V|\mu, R$ is IW distributed, due to the following reasoning:

$$\begin{aligned}
&f(V|(r_1,\ldots,r_T),\mu) \propto |\Omega V\Omega|^{-\frac{1}{2}}|V|^{-\frac{h+N+1}{2}}\exp\Big\{-\frac{1}{2}tr\left[V_0^{-1}V\right]\Big\}\\
&\times |V|^{-\frac{T}{2}}\exp\left[-\frac{1}{2}\sum_{t=1}^{T}(r_t-\mu)'V^{-1}(r_t-\mu)-\frac{1}{2}(\mu-\mu_0)'\frac{1}{\Omega}V^{-1}\frac{1}{\Omega}(\mu-\mu_0)\right]\\
&\propto |V|^{-\frac{2+N+T+h}{2}}\exp\Big\{-\frac{1}{2}tr\left[V_0V^{-1}\right]\Big\}\exp\Big\{-\frac{1}{2}tr\left[\left(\sum_t(r_t-\mu)(r_t-\mu)'\right)V^{-1}\right]\Big\}\\
&\times\exp\Big\{-\frac{1}{2}tr\left[\frac{1}{\Omega}(\mu-\mu_0)(\mu-\mu_0)'\frac{1}{\Omega}V^{-1}\right]\Big\}\\
&\propto |\rho V|^{-\frac{2+N+T+h}{2}}\exp\Big\{-\frac{1}{2}tr\left[\left(V_0+\sum_t(r_t-\mu)(r_t-\mu)'+\frac{1}{\Omega}(\mu-\mu_0)(\mu-\mu_0)'\frac{1}{\Omega}\right)V^{-1}\right]\Big\}
\end{aligned}$$

Hence, $V|\mu, R \sim IW(V_0+\sum_t(r_t-\mu)(r_t-\mu)'+\Omega^{-1}(\mu-\mu_0)(\mu-\mu_0)'\Omega^{-1}, T+h+1)$.

Given the conditional posterior distributions, $\mu|V, R$ and $V|\mu, R$, we implement the Gibbs sampling process to obtain the estimated values of the predictive means and covariances of the six assets employed in our study. To be more precise, after each sampling of μ, V, we sample predictive return from normal distribution with mean μ and covariance matrix V. Therefore, we can estimate the predictive density of R_{t+1}.

2 Robust Portfolio Choice

Following Hansen and Sargent (2001), Anderson, Hansen and Sargent (2003), and Anderson and Chen (2016), we allow agents to worry about model misspecification by considering perturbations to the probability density of asset returns that can decrease utility. Equivalently, people may cast doubt about a single probability law to describe the distribution of the relevant random variables. Although there are multiple ways to perturb distributions, we may assume that there is an adversarial agent acting against us. Given our portfolio choice, the adversarial agent perturbs the distribution to decrease our utility but incurs some perturbation cost. Meanwhile agents'fears about model specification doubts can be measured by ambiguity aversion. Under the adversarial agent setting, ambiguity aversion can be viewed as equivalently to adversarial agent's perturbation cost. If the adversarial agent has smaller cost to perturb distributions, they can perturb the distribution in a larger distribution space. Hence, the higher perturbation possibility investor should worry about or have higher level of ambiguity aversion. Taking the existence of this adversarial agent into consideration, investors choose their portfolio to maximize utility that is minimized by the adversary agent, or we are trying to solve a max-min or min-max question. This robust portfolio choice problem is similar spiritually to recent popular Generative Adversarial Networks in CS literature(Goodfellow et.al.(2014)).

According to investors' belief, we have a density function for future return R, let $z = R - E[R]$. Denote the p.d.f of z as $f(z)$, and perturbed distribution of z as $f(z)\rho(z)$. The robust portfolio choice problem can be set up as following:

$$\begin{aligned} \max_{\phi} \min_{\rho} \Bigg[\int_z \left[\phi' \left(\mu + z \right) - \frac{\gamma}{2} \left(\phi' z \right)^2 \right] f\left(z \right) \rho(z) dz \\ + \frac{1}{\tau} \int \rho\left(z \right) f\left(z \right) \log \rho\left(z \right) dz \Bigg] \end{aligned} \tag{7}$$

s.t.

$$\int \rho\left(z \right) f\left(z \right) dz = 1 \tag{8}$$

where the second term of equation 7 stands for the relative entropy(Kullback-Leibler divergence) between $\rho(z)f(z)$ and $f(z)$, measuring the discrepancy between the perturbed distribution and original one. And τ measures the ambiguity aversion level of investor, i.e. $1/\tau$ measures the cost of perturbing distribution.

Variational method and constraint 8 delivers the optimal condition for $\rho(z)$:

$$\varrho^{*}(z) = \frac{\exp\left[-\tau\phi' z + \frac{\theta\tau}{2}(\phi' z)^{2}\right]}{E\left(\exp\left[-\tau\phi' z + \frac{\theta\tau}{2}(\phi' z)^{2}\right]\right)} \tag{9}$$

Substituting equation 9 into equation 7, we can write the optimal portfolio choice problem as following

$$\max_{\phi}\left(\phi'\mu^{*} - \frac{1}{\tau}\log\int\exp\left[-\tau\phi' z + \frac{\theta\tau}{2}(\phi' z)^{2}\right]f(z)\,dz\right) \tag{10}$$

Though we have no analytical form for $f(z)$, which corresponds to predictive density of future returns, we can draw random sample from $f(z)$ and solve out the optimal portfolio choice according to equation 10.

Previous literature shows ambiguity aversion acts as an extra part of risk aversion(Trojani and Vanini (2004)). To offer an intuitive illustration for effects of ambiguity aversion and our following empirical results, we use Taylor expansions of equation 10. When our portfolio ϕ is close to zero, we have

$$\begin{aligned}
&\max_{\phi}\left(\phi'\mu^{*} - \frac{1}{\tau}\log\int\exp\left[-\tau\phi' z + \frac{\theta\tau}{2}(\phi' z)^{2}\right]f(z)\,dz\right)\\
&\approxeq \max_{\phi}\left(\phi'\mu^{*} - \frac{1}{\tau}\log\int(1 - \tau\phi' z + \frac{\tau(\theta+\tau)(\phi' z)^{2}}{2})f(z)dz\right)\\
&= \max_{\phi}\left(\phi'\mu^{*} - \frac{1}{\tau}\log(1 + \frac{\tau(\theta+\tau)\phi'\Sigma\phi}{2})\right)\\
&\approxeq \max_{\phi}\left(\phi'\mu^{*} - \frac{1}{\tau}(\frac{\tau(\theta+\tau)\phi'\Sigma\phi}{2})\right)\\
&= \max_{\phi}\left(\phi'\mu^{*} - (\frac{(\theta+\tau)\phi'\Sigma\phi}{2})\right)
\end{aligned}$$

where the second line follows Taylor expansion, omitting $o((\phi' z)^{2})$. The third line follows the definition of covariance matrix of z. The fourth line follows $\lim_{x\to 0}\frac{\log(1+x)}{x} = 1$. As one can tell from the final result, ambiguity aversion coefficient act as an extra term of risk aversion coefficient.

3 Robustness and Extensions

In this section, we present a number of additional analyses.

Figure 1 plots the continuous ex-ante certainty equivalent of returns gains from access to cryptocurrency, shown over time for various different priors. Panel A plots these for Bitcoin, while Panel B plots these for the equal-weighted cryptocurrency portfolio. These numbers are computed assuming that short sales are possible for Bitcoin after December 2017, but not possible for the equal-weighted portfolio. This is analogous to the results in Table 4 of the paper.

Figure 2 is similar, except that it plots the ex-post CER gains. It is the continuous version of Table 7 of the paper.

Figure 3 plots how the end-of-sample cutoff priors for non-investment vary according to the strength of priors. The latter is plotted in terms of the number of years of data the investor is presumed to have seen before the sample begins. Panel A shows this for Bitcoin, Panel B shows this for an equal-weighted cryptocurrency portfolio. These are the continuous version of Table 8 Panel A.

Figure 4 shows how end-of-sample cutoff mean priors for non-investment vary with the investor's priors about correlations between cryptocurrency and the equity market. Priors over correlations are considered between values of -0.5 and 0.5. Panel A shows the mean cutoff beliefs for Bitcoin, Panel B shows this for an equal-weighted cryptocurrency portfolio. These are the continuous versions of Table 8 Panel C.

Table 1 shows the priors required for non-investment in cryptocurrency at various points in the sample that represent local peaks and troughs of cryptocurrency prices. These are shown for Bitcoin (rows 1 and 2), equal-weighted cryptocurrency (rows 3 and 4), and value-weighted cryptocurrency (rows 5 and 6). Rows 1, 3 and 5 are priors for non-investment at that particular point in time, while rows 2, 4 and 6 are for zero or negative desired weights up to that point in time.

Table 2 presents the main results of the paper for the value-weighted cryptocurrency portfolio, analogous to those in the main text for the equal-weighted portfolio and bitcoin. Panel A presents summary statistics for the optimal portfolio weights at different points in time (the equivalent of Table 3). Panel B shows the ex-ante certainty equivalent gains at different points in time (the equivalent of Table 4). Panel C examines various levels of investment costs, and shows which priors are deterred from ever investing up to that point (the equivalent of Table 6 Panel A). Panel D shows a range a priors, and computes which costs would deter investment up to that point (the equivalent of Table 6 Panel C). Panel E shows the ex-post certainty equivalent gains from access to cryptocurrency at each point (the equivalent of Table 7).

Table 3 shows the ex-ante gains in Sharpe Ratios from having access to cryptocurrency, for various different priors, for bitcoin (Panel A), equal-weighted cryptocurrency (Panel B), and value-weighted cryptocurrency (Panel C).

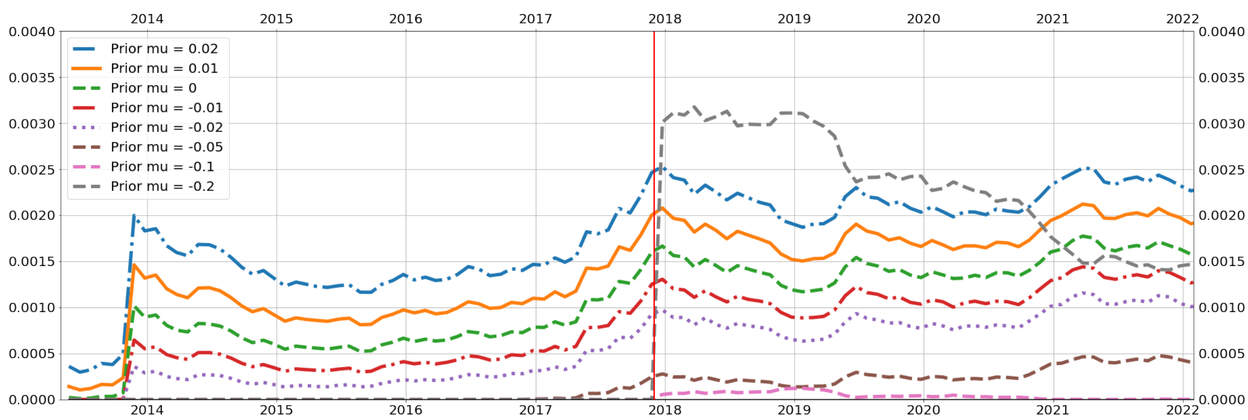
Table 4 shows how certainty equivalent gains vary with transaction costs. Panel A examines ex-ante CER gains, while Panel B examines ex-post CER gains. These are computed at the end of the sample for annual costs of 2%, 5%, 10% and 20%, for each cryptocurrency portfolio, and for the range of priors examined in other tables.

Table 5 presents the cutoff priors for non-investment under ambiguity aversion of $\tau = 4$, computed for the equal-weighted and value-weighted cryptocurrency portfolios. This is shown for non-investment at each time (Panel A), and no positive weights up to that point (Panel B). This is the equivalent of Table 8 Panel G.

Table 6 presents summary statistics for desired portfolio weights under ambiguity aversion of $\tau = 4$. Panel A shows equal-weighted cryptocurrency, and Panel B shows value-weighted cryptocurrency. This is the equivalent of Table 8 Panel H.

Table 7 shows how desired weights vary according to perceived volatility levels. Panels A and B examine the equal-weighted cryptocurrency portfolio, with Panel A showing average weights and Panel B showing end-of-sample weights. Panels C and D show the same results (average and end-of-sample respectively) for the value-weighted cryptocurrency portfolio. These results are the equivalent of Table 8 Panels D and E.

Figure 1: The Time Series of Ex-Ante Certainty Equivalent Gains from Cryptocurrency
This figure plots the time series of certainty equivalent of return (CER) gains from adding cryptocurrencies to investors' existing portfolios for different prior beliefs about the average monthly cryptocurrency return at the beginning of the sample period. The reported values equal the difference between the CER of the baseline market portfolio that excludes cryptocurrency and the CER of the optimal portfolio that combines the market portfolio and cryptocurrency, assuming that investors have a constant relative risk aversion of 3. We assume that investors can short Bitcoin starting December 2017, when the Chicago Mercantile Exchange (CME) introduced future contracts on Bitcoin, but cannot short the equal-weighted cryptocurrency portfolio throughout the sample period. Panel A shows the CER gains for Bitcoin, whereas Panel B shows the CER gains for the equal-weighted cryptocurrency portfolio. The sample consists of 106 monthly returns from May 2013 to February 2022. Panel A: Bitcoin



Panel B: Equal-weighted Cryptocurrency Portfolio

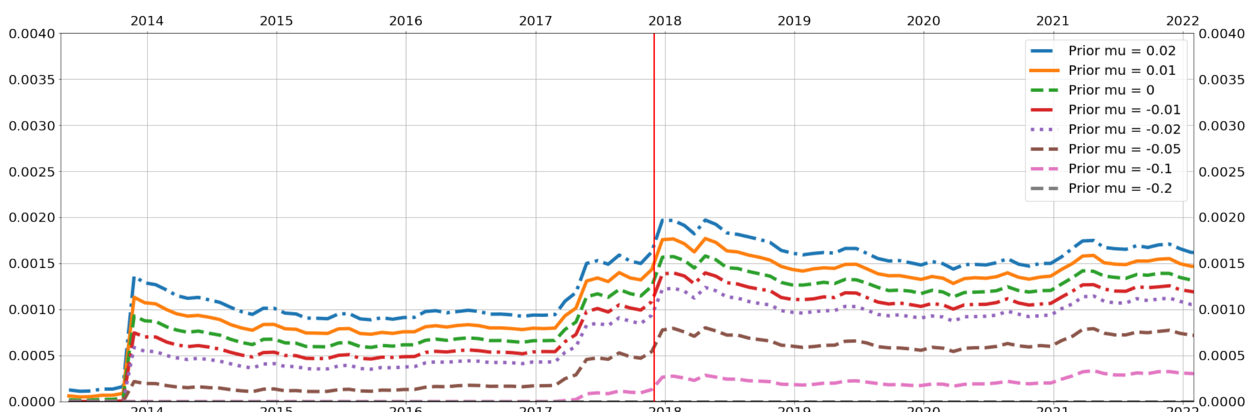
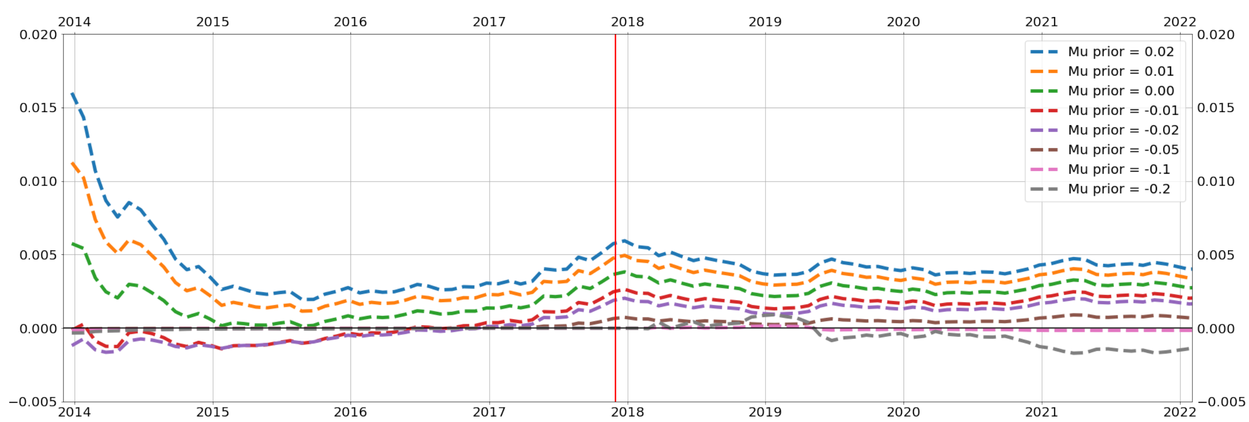


Figure 2: The Time Series of Ex-Post Certainty Equivalent Gains from Cryptocurrency
This figure plots the time series of ex-post certainty equivalent return (CER) gains from adding cryptocurrencies to investors' existing portfolios for different prior beliefs about the average monthly cryptocurrency return at the beginning of the sample period. Investors assess ex-post performance on a distributional basis, assuming that the distribution of realized returns up to that point (from whatever series of weights was chosen) were to continue indefinitely. The reported values equal the difference between the ex-post CER of the baseline market portfolio that excludes cryptocurrency and the ex-post CER of the optimal portfolio that combines the market portfolio and cryptocurrency. Investors are assumed to have a constant relative risk aversion of 3. We assume that investors can short Bitcoin starting December 2017, when the Chicago Mercantile Exchange (CME) introduced future contracts on Bitcoin, but cannot short the equal-weighted cryptocurrency portfolio throughout the sample period. Panel A shows the ex-post CER gains for Bitcoin, whereas Panel B shows the ex-post CER gains for the equal-weighted cryptocurrency portfolio. The sample consists of 106 monthly returns from May 2013 to February 2022.

Panel A: Bitcoin



Panel B: Equal-weighted Cryptocurrency Portfolio

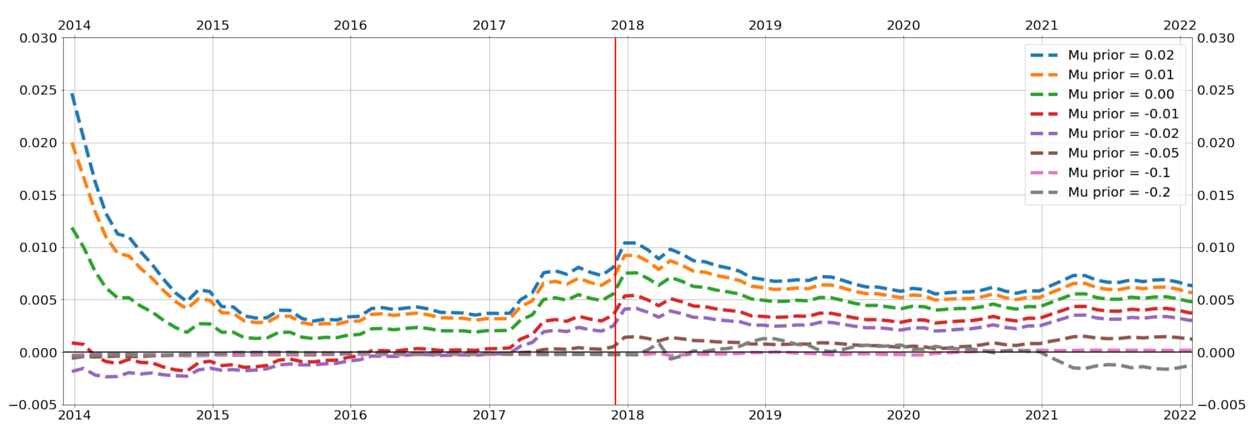
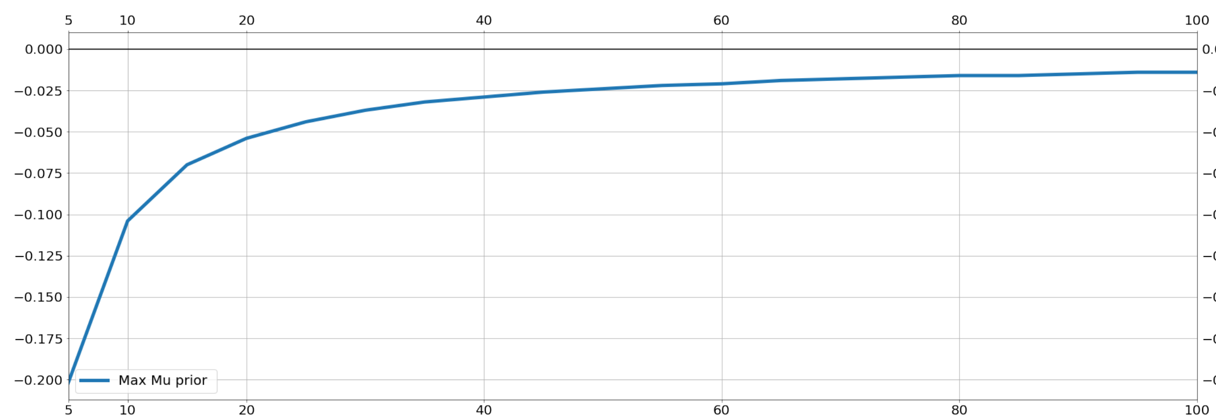


Figure 3: The Strength of Prior Beliefs and Non-Investment in Cryptocurrency
This figure plots the cutoff prior beliefs about the average monthly cryptocurrency return at the beginning of the sample period required for non-investment in cryptocurrency by the end of the sample period, as a function of the strength of the prior (ranging from 5 to 100 years of pre-sample data observed by investors). Panel A shows the cutoff beliefs for Bitcoin, whereas Panel B shows the cutoff beliefs for the equal-weighted cryptocurrency portfolio. The sample consists of 106 monthly returns from May 2013 to February 2022. Panel A: Bitcoin



Panel B: Equal-weighted Cryptocurrency Portfolio

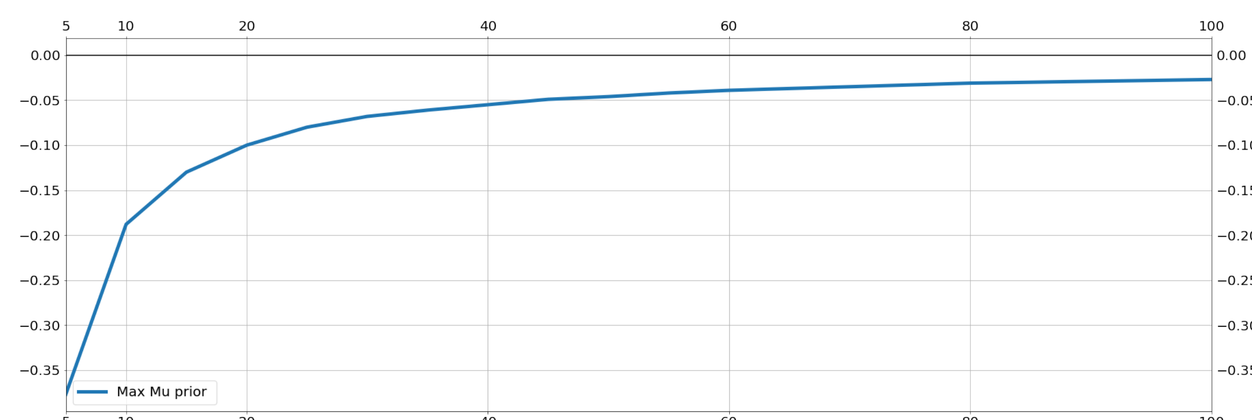
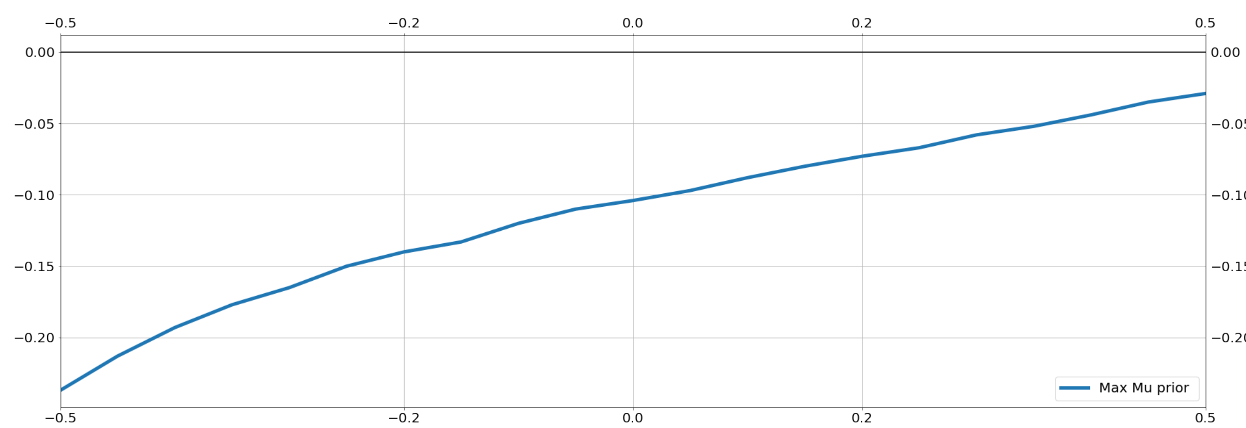


Figure 4: Prior Beliefs about Correlations and Non-Investment in Cryptocurrency
This figure plots the cutoff prior beliefs about the average monthly cryptocurrency return at the beginning of the sample period required for non-investment in cryptocurrency by the end of the sample period, as a function of prior beliefs about the correlation between cryptocurrencies and the market portfolio. Panel A shows the cutoff beliefs for Bitcoin, whereas Panel B shows the cutoff beliefs for the equal-weighted cryptocurrency portfolio. The sample consists of 106 monthly returns from May 2013 to February 2022. Panel A: Bitcoin



Panel B: Equal-weighted Cryptocurrency Portfolio

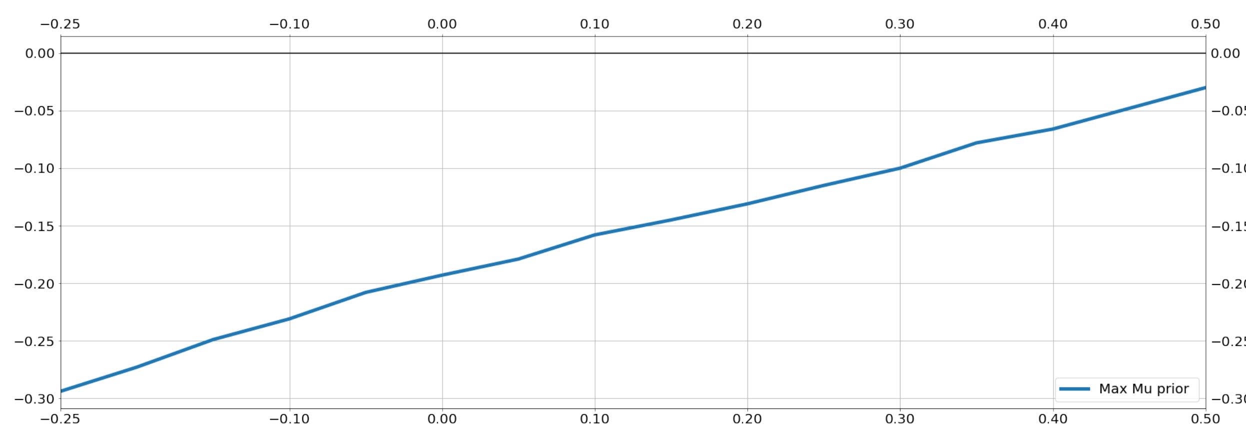


Table 1: Peaks and Troughs

This table reports cutoff prior beliefs about the average monthly cryptocurrency return at the beginning of the sample period that would make an investor not invest in cryptocurrencies. Each row corresponds to a different trough or peak months during the sample period, and the columns correspond to different cryptocurrency portfolios (Bitcoin, Equal-weighted cryptocurrency portfolio, Value-weighted cryptocurrency portfolio). The odd columns calculate the cutoff prior belief that leads to no cryptocurrency investment on a specific date, whereas the even columns calculate the cutoff belief that leads to no cryptocurrency investment at any point prior to the date. If the priors are above (below) the cut-ff level, then investors should long (short) on a specific date (odd columns) or at some point prior to the date (even columns). The calculations assume the following: (1) Investors start with the CRSP value-weighted market portfolio as a base asset and consider adding cryptocurrencies to their portfolios; (2) Investors observed ten years of data with a mean equal to their prior mean before the beginning of the sample period; (3) The variance of cryptocurrency returns approximately equals their ex-post variance – 150 times the market variance for bitcoin, 170 times the market variance for the value-weighted cryptocurrency portfolio, and 625 times the market variance for the equal-weighted portfolio; (4) Investors believe cryptocurrency to be uncorrelated with the market portfolio. The sample consists of 106 monthly returns from May 2013 to February 2022.

	BTC		EW		VW	
	Snap-shot	Cumu-lative	Snap-shot	Cumu-lative	Snap-shot	Cumu-lative
Jun-13 (2013 Trough)	-0.005	-0.007	-0.011	-0.013	-0.005	-0.007
Sep-13 (2013 Local Trough)	-0.009	-0.009	-0.015	-0.015	-0.009	-0.009
Nov-13 (2013-2014 Peak)	-0.050	-0.050	-0.097	-0.097	-0.052	-0.052
Apr-14 (Mt Gox Trough)	-0.044	-0.050	-0.091	-0.097	-0.046	-0.052
Jun-14 (2014 Peak)	-0.047	-0.050	-0.092	-0.097	-0.049	-0.052
Jan-15 (2014-2015 Trough)	-0.040	-0.050	-0.088	-0.097	-0.042	-0.052
Aug-15 (2015 Local Trough)	-0.041	-0.050	-0.090	-0.097	-0.043	-0.052
Jan-16 (2016 Trough)	-0.046	-0.050	-0.093	-0.097	-0.047	-0.052
Jun-16 (2016 Local Trough)	-0.051	-0.051	-0.102	-0.102	-0.053	-0.053
Aug-16 (2016 Local Trough)	-0.050	-0.051	-0.100	-0.102	-0.051	-0.053
Mar-17 (2017 Local Trough)	-0.056	-0.058	-0.113	-0.113	-0.059	-0.059
Dec-17 (Pre-2017 Peak)	-0.085	-0.085	-0.171	-0.171	-0.091	-0.091
Jan-19 (19-20 Trough)	-0.076	-0.085	-0.161	-0.175	-0.080	-0.091
Jun-19 (2019 Peak)	-0.090	-0.090	-0.169	-0.175	-0.091	-0.091
Mar-20 (2020 Trough)	-0.084	-0.090	-0.163	-0.175	-0.087	-0.091
Sep-20 (2020 Local Trough)	-0.088	-0.090	-0.171	-0.175	-0.091	-0.092
Mar-21 (2021 Local Peak)	-0.103	-0.103	-0.192	-0.192	-0.106	-0.106
May-21 (2021 Local Trough)	-0.100	-0.103	-0.191	-0.193	-0.105	-0.107
Aug-21 (2021 Local Peak)	-0.102	-0.103	-0.194	-0.194	-0.108	-0.108
Sep-21 (2021 Local Trough)	-0.102	-0.103	-0.193	-0.194	-0.107	-0.108
Oct-21 Peak	-0.106	-0.106	-0.195	-0.195	-0.110	-0.110
Jan-22 (2022 Local Trough)	-0.102	-0.106	-0.192	-0.196	-0.106	-0.110
Feb-22 (End of Sample)	-0.103	-0.106	-0.192	-0.196	-0.107	-0.110

Table 2: The Value-Weighted Cryptocurrency portfolio

This table re-estimates the main analyses for the value-weighted cryptocurrency portfolio. Panel A reports cryptocurrency portfolio weights for different prior beliefs about the average monthly cryptocurrency excess return at the beginning of the sample period. Panel B reports the monthly certainty equivalent return (CER) gains, in percentage points, from adding cryptocurrency to investors' existing portfolios. Panel C reports the range of prior means that would justify noninvestment in cryptocurrencies throughout the sample period for different investment costs (in percent per year, applied to the absolute value of the weight in cryptocurrency). Panel D reports the inverse - for each of the priors in question, how high would costs have to be to result in non-investment up to that point? panel E reports investors' certainty equivalent return (CER) gains for the ex-post distribution of portfolio returns relative to the ex-post distribution of the market portfolio over the same period. Investors are assumed to have a relative risk aversion of 3. The sample consists of 106 monthly returns from May 2013 to February 2022.

Panel A: Optimal Portfolio Weights

Prior	Average	S.t.d	Lowest	Highest	Final	Fraction of positive	Fraction of above 0.5%	Fraction of above 1%	Fraction of above 2%	Fraction of above 5%	First Date Weight is Positive	Mean of Leverage	S.t.d of Leverage
Flat	0.146	0.084	-0.037	0.588	0.109	0.991	0.991	0.991	0.991	0.962	2013-06	1.082	0.082
2	0.053	0.010	0.022	0.066	0.063	1.000	1.000	1.000	1.000	0.613	all above	0.989	0.036
1	0.047	0.011	0.013	0.061	0.058	1.000	1.000	1.000	0.943	0.538	all above	0.979	0.035
0	0.041	0.012	0.004	0.056	0.053	1.000	0.991	0.953	0.943	0.217	all above	0.975	0.041
-1	0.034	0.013	-0.005	0.05	0.048	0.953	0.943	0.943	0.943	0.019	2013-10	0.968	0.045
-2	0.028	0.014	-0.014	0.045	0.043	0.943	1.000	0.991	0.726	0.000	2013-11	0.963	0.046
-5	0.009	0.016	-0.04	0.03	0.028	0.670	0.679	0.604	0.434	0.000	2013-11	0.943	0.047
-10	-0.022	0.021	-0.085	0.005	0.003	0.123	0.830	0.557	0.453	0.057	2021-02	0.910	0.051
-20	-0.084	0.031	-0.175	-0.046	-0.047	0.000	1.000	1.000	1.000	0.877	all below	0.853	0.055

Panel B: Ex-ante Certainty Equivalent Gains

Prior	2013	2014	2015	2016	2017	2018	2019	2020	2021	End of Sample
Flat	16.143	2.221	1.093	0.825	1.435	0.775	0.654	0.699	0.678	0.634
2	0.173	0.126	0.124	0.134	0.248	0.184	0.182	0.209	0.219	0.211
1	0.127	0.090	0.090	0.100	0.205	0.149	0.149	0.176	0.185	0.178
0	0.087	0.059	0.061	0.072	0.166	0.118	0.120	0.145	0.155	0.149
-1	0.055	0.036	0.038	0.048	0.131	0.091	0.094	0.116	0.128	0.122
-2	0.030	0.018	0.021	0.029	0.101	0.067	0.071	0.091	0.101	0.098
-5	0.000	0.001	0.000	0.001	0.034	0.017	0.021	0.033	0.043	0.042
-10	0.093	0.096	0.072	0.046	0.002	0.007	0.003	0.000	0.001	0.001
-20	0.808	0.737	0.609	0.478	0.242	0.257	0.208	0.157	0.118	0.118

Panel C: Investment Costs and Beliefs Required for Non-Investment

Cost/Time	2013	2014	2015	2016	2017	2018	2019	2020	2021	End of Sample
10 (min)	nan	nan	nan	nan	nan	nan	nan	nan	nan	nan
(max)	nan	nan	nan	nan	nan	nan	nan	nan	nan	nan
15 (min)	-2.90	-2.90	-2.90	-2.90	nan	nan	nan	nan	nan	nan
(max)	-2.60	-2.60	-2.60	-2.60	nan	nan	nan	nan	nan	nan
20 (min)	-3.80	-3.80	-3.80	-3.80	nan	nan	nan	nan	nan	nan
(max)	-1.70	-1.70	-1.70	-1.70	nan	nan	nan	nan	nan	nan
30 (min)	-5.50	-5.50	-5.50	-5.50	-5.50	-5.50	-5.50	-5.50	-5.50	-5.50
(max)	0.10	0.10	0.10	0.10	-1.70	-1.70	-1.70	-1.70	-1.80	-1.80
50 (min)	-8.90	-8.90	-8.90	-8.90	-8.90	-8.90	-8.90	-8.90	-8.90	-8.90
(max)	3.60	3.60	3.60	3.60	3.20	3.20	3.20	3.20	3.20	3.20
Lowest Preventing Cost	13.80	13.80	13.80	13.80	20.90	20.90	20.90	20.90	22.00	22.00
Corresponding Prior	-2.80	-2.80	-2.80	-2.80	-4.00	-4.00	-4.00	-4.00	-4.10	-4.10

Panel D: Minimum Cost for Non-Investment (in Percent)

Prior	2013	2014	2015	2016	2017	2018	2019	2020	2021	End of Sample
2	40.64	40.64	40.64	40.64	45.08	45.08	45.08	45.08	45.08	45.08
1	35.05	35.05	35.05	35.05	41.00	41.00	41.00	41.00	41.00	41.00
0	29.42	29.42	29.42	29.42	36.83	36.83	36.83	36.83	36.83	36.83
-1	23.80	23.80	23.80	23.80	32.76	32.76	32.76	32.76	32.76	32.76
-2	18.19	18.19	18.19	18.19	28.74	28.74	28.74	28.74	28.97	28.97
-5	1.00	1.00	1.00	2.31	16.63	16.63	16.63	16.63	19.07	19.07
-10	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	3.25	3.25
-20	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

Panel E: Ex-post Certainty Equivalent Gains

Prior	2013	2014	2015	2016	2017	2018	2019	2020	2021	End of Sample
Flat	-1.633	-2.288	-1.386	-0.763	0.197	-0.171	-0.053	0.078	0.178	0.15
2	1.519	0.347	0.241	0.266	0.579	0.338	0.334	0.362	0.381	0.362
1	1.061	0.219	0.161	0.195	0.483	0.274	0.276	0.304	0.325	0.308
0	0.539	0.065	0.063	0.111	0.378	0.202	0.209	0.24	0.264	0.249
-1	-0.032	-0.111	-0.049	0.017	0.263	0.122	0.137	0.171	0.197	0.184
-2	-0.67	-0.317	-0.179	-0.089	0.138	0.032	0.056	0.093	0.123	0.113
-5	-2.953	-1.089	-0.666	-0.482	-0.297	-0.285	-0.229	-0.174	-0.129	-0.131
-10	-7.996	-2.902	-1.809	-1.375	-1.219	-0.983	-0.848	-0.747	-0.662	-0.649
-20	-22.642	-8.475	-5.317	-4.052	-3.798	-3.003	-2.624	-2.358	-2.143	-2.092
Max Gain	2.933	0.583	0.393	0.438	0.981	0.543	0.54	0.587	0.624	0.587
Optimal Prior	0.085	0.061	0.062	0.074	0.111	0.088	0.092	0.104	0.111	0.111
Positive CER, max prior	0.183	0.127	0.13	0.157	0.247	0.201	0.216	0.238	0.259	0.254
Positive CER, min prior	-0.009	-0.003	-0.005	-0.011	-0.03	-0.023	-0.026	-0.031	-0.035	-0.034

Table 3: Ex-ante Sharp Ratio Gains

This table reports estimates of the perceived monthly gain in Sharpe Ratios, in percentage points, from adding cryptocurrency to investors' existing portfolios. Panel A is for Bitcoin, Panel B is for an equal-weighted portfolio of cryptocurrency, and Panel C is for a value-weighted portfolio of cryptocurrency The reported values equal the difference between the Sharpe ratio of the baseline market portfolio that excludes cryptocurrency and the Sharpe ratio of the optimal portfolio that combines the market portfolio and cryptocurrency, assuming that investors have a constant relative risk aversion of 3. Years correspond to the end of the calendar year in question. Each row corresponds to a different prior belief, and each column corresponds to a specific end-of-year date when the investment decision is made. The sample consists of 106 monthly returns from May 2013 to February 2022.

Panel A: Ex-ante Sharpe Ratio Gains for BTC in Percentage

Prior	2013	2014	2015	2016	2017	2018	2019	2020	2021	End of Sample
Flat	88.009	26.423	16.980	13.930	18.907	12.867	11.776	12.671	11.106	10.339
2	3.674	2.675	2.877	3.009	5.035	3.953	4.105	4.404	4.440	4.304
1	2.714	1.937	2.111	2.338	4.219	3.210	3.469	3.903	3.743	3.706
0	1.910	1.317	1.494	1.687	3.507	2.557	2.852	3.274	3.182	3.241
-1	1.227	0.773	0.952	1.182	2.838	1.954	2.312	2.715	2.614	2.637
-2	0.658	0.371	0.506	0.720	2.143	1.490	1.768	2.199	2.143	2.158
-5	0.011	0.057	0.007	0.014	0.636	0.330	0.551	0.852	0.913	0.848
-10	2.437	2.478	1.835	1.249	0.130	0.283	0.102	0.003	0.004	0.002
-20	14.044	13.239	11.442	9.721	5.876	6.337	4.831	3.332	2.871	2.880

Panel B: Ex-ante Sharpe Ratio Gains for EW in Percentage

Prior	2013	2014	2015	2016	2017	2018	2019	2020	2021	End of Sample
Flat	86.858	27.958	16.182	12.173	18.269	12.911	10.277	9.521	9.875	9.276
2	2.681	2.141	1.965	2.038	3.984	3.288	2.995	3.035	3.274	3.321
1	2.271	1.807	1.654	1.736	3.662	2.971	2.725	2.729	2.929	3.029
0	1.880	1.450	1.379	1.452	3.340	2.655	2.453	2.386	2.676	2.695
-1	1.528	1.164	1.101	1.201	2.997	2.382	2.212	2.134	2.508	2.428
-2	1.202	0.901	0.863	0.981	2.694	2.106	1.978	1.911	2.269	2.158
-5	0.443	0.306	0.305	0.394	1.769	1.365	1.222	1.253	1.569	1.473
-10	0.006	0.016	0.010	0.000	0.613	0.452	0.401	0.442	0.633	0.635
-20	2.277	2.110	1.786	1.333	0.111	0.167	0.140	0.067	0.003	0.005

Panel C: Ex-ante Sharpe Ratio Gains for VW in Percentage

Prior	2013	2014	2015	2016	2017	2018	2019	2020	2021	End of Sample
Flat	86.676	26.154	16.027	12.772	19.154	12.159	10.818	11.051	10.476	10.073
2	3.567	2.619	2.585	2.764	4.976	3.865	3.715	4.101	4.091	4.096
1	2.680	1.910	1.939	2.125	4.237	3.235	3.113	3.480	3.525	3.531
0	1.880	1.286	1.343	1.525	3.454	2.672	2.540	2.984	3.028	3.036
-1	1.212	0.801	0.872	1.034	2.809	2.112	2.070	2.448	2.558	2.541
-2	0.691	0.405	0.482	0.643	2.182	1.574	1.592	1.949	2.040	2.038
-5	0.001	0.024	0.002	0.015	0.752	0.409	0.509	0.704	0.873	0.901
-10	2.031	2.070	1.596	1.030	0.048	0.166	0.084	0.003	0.020	0.013
-20	12.575	11.621	10.323	8.576	4.778	5.321	4.306	3.017	2.367	2.379

Table 4: Investment Costs and Certainty Equivalent Return Gains

This table considers the effect of annual investment costs on certainty equivalent return (CER) gains from cryptocurrency. Panel A reports the ex-ante monthly CER gain, in percentage points, from adding cryptocurrency to investors' existing portfolios, which corresponds to the end of the sample period. Panel B reports analogous estimates of ex-post monthly CER gains. Each row corresponds to a different prior belief. Each column corresponds to a different combination of a cryptocurrency asset and an annual investment cost (in percent per year, applied to the absolute value of the weight in cryptocurrency). Investors are assumed to have a relative risk aversion of 3. The sample consists of 106 monthly returns from May 2013 to February 2022.

Panel A: Ex-ante Gains

	Annual cost 20%	Annual cost 10%	Annual cost 5%	Annual cost 2%
BTC				
Flat	0.479	0.579	0.629	0.660
2	0.112	0.171	0.200	0.218
1	0.086	0.140	0.166	0.183
0	0.063	0.112	0.137	0.151
-1	0.042	0.086	0.108	0.121
-2	0.024	0.064	0.083	0.095
-5	-0.008	0.016	0.029	0.036
-10	-0.002	-0.001	-0.001	0.000
-20	0.051	0.097	0.119	0.133
EW				
Flat	0.464	0.508	0.530	0.544
2	0.113	0.137	0.149	0.157
1	0.101	0.124	0.135	0.142
0	0.088	0.109	0.120	0.127
-1	0.076	0.097	0.107	0.113
-2	0.066	0.086	0.095	0.101
-5	0.039	0.055	0.063	0.068
-10	0.009	0.020	0.025	0.028
-20	-0.002	-0.001	0.000	0.000
VW				
Flat	0.452	0.543	0.588	0.616
2	0.106	0.158	0.185	0.201
1	0.082	0.130	0.154	0.169
0	0.060	0.105	0.127	0.140
-1	0.042	0.082	0.102	0.114
-2	0.027	0.063	0.080	0.091
-5	-0.005	0.019	0.031	0.038
-10	-0.005	-0.002	-0.001	0.000
-20	0.040	0.079	0.099	0.111

Panel B: Ex-post Gains

	Annual cost 20%	Annual cost 10%	Annual cost 5%	Annual cost 2%
BTC				
Flat	-0.072	0.063	0.131	0.172
2	0.306	0.355	0.38	0.395
1	0.258	0.301	0.323	0.336
0	0.203	0.240	0.259	0.270
-1	0.142	0.174	0.190	0.199
-2	0.074	0.100	0.114	0.122
-5	-0.174	-0.161	-0.154	-0.15
-10	-0.771	-0.749	-0.738	-0.731
-20	-2.512	-2.433	-2.393	-2.369
EW				
Flat	0.283	0.347	0.379	0.398
2	0.248	0.269	0.279	0.285
1	0.221	0.24	0.25	0.256
0	0.192	0.209	0.218	0.224
-1	0.161	0.177	0.185	0.190
-2	0.128	0.143	0.151	0.155
-5	0.019	0.031	0.036	0.040
-10	-0.194	-0.188	-0.185	-0.183
-20	-0.768	-0.757	-0.752	-0.749
VW				
Flat	-0.097	0.027	0.089	0.126
2	0.272	0.317	0.339	0.353
1	0.229	0.269	0.288	0.300
0	0.181	0.215	0.232	0.242
-1	0.126	0.155	0.17	0.178
-2	0.064	0.088	0.101	0.108
-5	-0.155	-0.143	-0.137	-0.133
-10	-0.686	-0.667	-0.658	-0.652
-20	-2.231	-2.162	-2.127	-2.106

Table 5: Snapshot and Cumulative Cutoff Priors for the Equal-Weighted and Value-Weighted Cryptocurrency Portfolios, with ambiguity aversion $\tau = 4$
This table reports the snapshot and accumulative cutoff priors for non-investment in the equal-weighted and Value-weighted cryptocurrency portfolio in a robust portfolio choice framework.

Panel A: Snapshot Cutoff Priors

	2013	2014	2015	2016	2017	2018	2019	2020	2021	End of Sample
ew-rf	-0.095	-0.091	-0.093	-0.102	-0.17	-0.163	-0.164	-0.172	-0.196	-0.193
vw-rf	-0.049	-0.044	-0.049	-0.056	-0.091	-0.081	-0.086	-0.098	-0.109	-0.107

Panel B: Cumulative Cutoff Priors

	2013	2014	2015	2016	2017	2018	2019	2020	2021	End of Sample
ew-rf	-0.097	-0.097	-0.097	-0.102	-0.170	-0.175	-0.175	-0.175	-0.197	-0.197
vw-rf	-0.052	-0.052	-0.052	-0.056	-0.091	-0.091	-0.091	-0.098	-0.111	-0.111

Table 6: Optimal Portfolio Weights for the Equal-Weighted and Value-Weighted Cryptocurrency Portfolios, with ambiguity aversion $\tau = 4$

This table reports cryptocurrency portfolio weights for different prior beliefs about the average monthly cryptocurrency excess return at the beginning of the sample period, with ambiguity aversion $\tau = 4$. Panel A considers an equal-weighted cryptocurrency portfolio, and Panel B considers a value-weighted cryptocurrency portfolio. Each row corresponds to a different prior belief. For each prior, the columns indicate a range of attributes of the distribution of weights over the sample period - the average, lowest, highest, final (i.e. end-of-sample) weight, the fraction of months that are above zero, the fraction of weights whose absolute value exceeds 0.5%, 1.5%, 2.5% and 5%, the first date in the sample when the weight is positive, as well as the mean and standard deviation of leverage choices. The sample consists of 106 monthly returns from May 2013 to February 2022.

Panel A: Equal-Weighted Cryptocurrency Portfolio

Prior	Average	S.t.d	Lowest	Highest	Final	Fraction positive	Fraction above 0.5%	Fraction above 1%	Fraction above 2%	Fraction above 5%	First Date Weight is Positive	Mean of Leverage	S.t.d of Leverage
2	0.010	0.002	0.003	0.013	0.012	1.000	0.943	0.585	0.000	0.000	All above	0.411	0.013
1	0.010	0.002	0.002	0.012	0.012	1.000	0.943	0.547	0.000	0.000	All above	0.41	0.013
0	0.009	0.002	0.001	0.012	0.011	1.000	0.943	0.519	0.000	0.000	All above	0.409	0.013
-1	0.008	0.003	0.000	0.011	0.010	1.000	0.943	0.292	0.000	0.000	All above	0.407	0.012
-2	0.007	0.003	-0.001	0.010	0.010	0.943	0.943	0.113	0.000	0.000	2013-Nov	0.406	0.011
-5	0.005	0.003	-0.004	0.008	0.008	0.943	0.557	0.000	0.000	0.000	2014-Nov	0.404	0.011
-10	0.002	0.004	-0.009	0.006	0.005	0.632	0.179	0.000	0.000	0.000	2016-Jun	0.398	0.014
-20	-0.005	0.005	-0.019	0.000	0.000	0.000	0.453	0.057	0.000	0.000	All Below	0.396	0.016

Panel B: Value-Weighted Cryptocurrency Portfolio

Prior	Average	S.t.d	Lowest	Highest	Final	Fraction positive	Fraction above 0.5%	Fraction above 1%	Fraction above 2%	Fraction above 5%	First Date Weight is Positive	Mean of Leverage	S.t.d of Leverage
2	0.023	0.004	0.009	0.028	0.027	1.000	1.000	0.981	0.736	0.000	All above	0.419	0.014
1	0.020	0.005	0.006	0.026	0.025	1.000	1.000	0.943	0.557	0.000	All above	0.416	0.015
0	0.017	0.005	0.002	0.024	0.023	1.000	0.943	0.943	0.425	0.000	All above	0.414	0.015
-1	0.015	0.005	-0.002	0.021	0.020	0.953	0.943	0.868	0.142	0.000	2013-Oct	0.411	0.018
-2	0.012	0.006	-0.006	0.019	0.018	0.943	0.972	0.575	0.000	0.000	2013-Nov	0.409	0.019
-5	0.004	0.007	-0.017	0.013	0.012	0.670	0.604	0.217	0.000	0.000	2014-Nov	0.402	0.020
-10	-0.009	0.009	-0.036	0.002	0.001	0.132	0.509	0.453	0.057	0.000	2021-Jan	0.387	0.023
-20	-0.035	0.013	-0.072	-0.019	-0.020	0.000	1.000	1.000	0.934	0.057	All Below	0.357	0.026

Table 7: Optimal Weights of the Equal-Weighted and Value-Weighted Cryptocurrency Portfolios for Different Volatility Priors

This table explores how different prior beliefs about volatility affect the average and end-of-sample weights on the equal-weighted and value-weighted cryptocurrency Portfolios across different prior means Panel A: Average Weights of the Equal-Weighted Cryptocurrency Portrolio

Volatility \EW prior	2	1	0	-1	-2	-5	-10	-20
0.2x Sample	0.147	0.133	0.12	0.107	0.093	0.054	-0.012	-0.139
0.5x Sample	0.068	0.063	0.058	0.053	0.048	0.032	0.007	-0.043
Baseline	0.024	0.023	0.021	0.019	0.018	0.013	0.004	-0.013
2x Sample	0.007	0.007	0.006	0.006	0.005	0.004	0.002	-0.003
5x Sample	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.000

Panel B: End-of-Sample Weights of the Equal-Weighted Cryptocurrency Portrolio

Volatility \EW prior	2	1	0	-1	-2	-5	-10	-20
0.2x Sample	0.160	0.152	0.143	0.136	0.127	0.103	0.063	-0.014
0.5x Sample	0.079	0.075	0.071	0.067	0.063	0.052	0.032	-0.006
Baseline	0.029	0.027	0.026	0.025	0.023	0.019	0.012	-0.001
2x Sample	0.009	0.008	0.008	0.008	0.007	0.006	0.004	0.000
5x Sample	0.002	0.002	0.002	0.002	0.001	0.001	0.001	0.000

Panel C: Average Weights of the Value-Weighted Cryptocurrency Portrolio

Volatility \VW prior	2	1	0	-1	-2	-5	-10	-20
0.2x Sample	0.340	0.290	0.239	0.189	0.139	-0.009	-0.249	-0.699
0.5x Sample	0.151	0.132	0.113	0.094	0.075	0.018	-0.076	-0.257
Baseline	0.053	0.047	0.041	0.034	0.028	0.009	-0.022	-0.084
2x Sample	0.016	0.014	0.012	0.011	0.009	0.004	-0.005	-0.022
5x Sample	0.003	0.003	0.003	0.002	0.002	0.001	0.000	-0.003

Panel D: End-of-Sample Weights of the Value-Weighted Cryptocurrency Portrolio

Volatility \VW prior	2	1	0	-1	-2	-5	-10	-20
0.2x Sample	0.351	0.322	0.292	0.262	0.232	0.144	0.001	-0.257
0.5x Sample	0.175	0.161	0.146	0.131	0.117	0.074	0.004	-0.129
Baseline	0.063	0.058	0.053	0.048	0.043	0.028	0.003	-0.047
2x Sample	0.019	0.018	0.016	0.015	0.014	0.009	0.002	-0.012
5x Sample	0.004	0.004	0.003	0.003	0.003	0.002	0.001	-0.001